

# Examples of groups which are not weakly amenable

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**ABSTRACT.** We prove that weak amenability of a locally compact group imposes a strong condition on its amenable closed normal subgroups. This extends non weak amenability results of Haagerup (1988) and Ozawa–Popa (2010). A von Neumann algebra analogue is also obtained.

## 1. INTRODUCTION

Let  $G$  be a group, which is always assumed to be a locally compact topological group. The group  $G$  is said to be *weakly amenable* if the Fourier algebra  $\mathcal{A}G$  of  $G$  has an approximate identity  $(\varphi_n)$  which is uniformly bounded as Herz–Schur multipliers. (If one requires  $(\varphi_n)$  to be bounded as elements in  $\mathcal{A}G$ , it becomes one of the equivalent definitions of amenability.) See Section 2 for the precise definition. Weak amenability is strictly weaker than amenability and passes to closed subgroups. It is proved by De Cannière–Haagerup, Cowling and Cowling–Haagerup ([dCH, Co, CH]) that real simple Lie groups of real rank one are weakly amenable (see also [Oz]), and by Haagerup ([Ha]) that real simple Lie groups of real rank at least two are not weakly amenable. For the latter fact, Haagerup proves that  $\mathrm{SL}(2, \mathbb{R}) \rtimes \mathbb{R}^2$  is not weakly amenable. (See also [Do].) More recently, it is proved by Ozawa–Popa ([OP]) that the wreath product  $\Lambda \wr \Gamma$  of a non-trivial group  $\Lambda$  by a non-amenable discrete group  $\Gamma$  is not “weakly amenable with constant 1.” In this paper, we generalize these non weak amenability results as follows.

**Theorem A.** *Let  $G$  be an weakly amenable group and  $N$  be an amenable closed normal subgroup of  $G$ . Then, there is a  $G \rtimes N$ -invariant state on  $L^\infty(N)$ , where the semidirect product  $G \rtimes N$  acts on  $N$  by  $(g, a) \cdot x = gaxg^{-1}$ .*

In particular, the wreath product by a non-amenable group is never weakly amenable. The theorem also gives a new proof of Haagerup’s result that  $\mathrm{SL}(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$  is not weakly amenable, without appealing to the lattice embedding into  $\mathrm{SL}(2, \mathbb{R}) \rtimes \mathbb{R}^2$ . We note for the sake of completeness that there is an even weaker variant of weak amenability, called the *approximation property* ([HK]), and  $\mathrm{SL}(2, \mathbb{R}) \rtimes \mathbb{R}^2$  has the approximation property, while  $\mathrm{SL}(n \geq 3, \mathbb{R})$  does not ([LdS]).

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As Theorem 3.5 in [OP], there is an analogous result for von Neumann algebras. We refer to Section 3 in [OP] and Section 4 of this paper for the terminology used in the following theorem.

**Theorem B.** *Let  $M$  be a finite von Neumann algebra with the weak\* completely bounded approximation property. Then, every amenable von Neumann subalgebra  $P$  is weakly compact in  $M$ .*

It follows that a type II<sub>1</sub> factor having the weak\* completely bounded approximation property and property (T) (e.g., the group von Neumann algebra of a torsion-free lattice in  $\mathrm{Sp}(1, n)$ ) is not isomorphic to a group-measure-space von Neumann algebra.

## 2. PRELIMINARY ON HERZ–SCHUR MULTIPLIERS

Let  $G$  be a group. We denote by  $\lambda$  the left regular representation of  $G$  on  $L^2(G)$ , by  $C_\lambda^*G$  the reduced group C\*-algebra and by  $\mathcal{L}G$  the group von Neumann algebra of  $G$ . The *Fourier algebra*  $\mathcal{A}G$  of  $G$  consists of all functions  $\varphi$  on  $G$  such that there are vectors  $\xi, \eta \in L^2(G)$  satisfying  $\varphi(x) = \langle \lambda(x)\xi, \eta \rangle$  for every  $x \in G$ . (In other words,  $\mathcal{A}G = L^2(G) * L^2(G)$ .) It is a Banach algebra with the norm  $\|\varphi\| = \inf\{\|\xi\|_\infty\|\eta\|_\infty\}$ , where the infimum is taken over all  $\xi, \eta \in L^2(G)$  as above. The Fourier algebra  $\mathcal{A}G$  is naturally identified with the predual of  $\mathcal{L}G$  under the duality pairing  $\langle \varphi, \lambda(f) \rangle = \int_G \varphi f$  for  $\varphi \in \mathcal{A}G$  and  $\lambda(f) \in \mathcal{L}G$ . If  $H$  is a closed subgroup of  $G$ , then  $\varphi|_H \in \mathcal{A}H$  for every  $\varphi \in \mathcal{A}G$ . A continuous function  $\varphi$  on  $G$  is called a *Herz–Schur multiplier* if there are a Hilbert space  $\mathcal{H}$  and bounded continuous functions  $\xi, \eta: G \rightarrow \mathcal{H}$  such that  $\varphi(y^{-1}x) = \langle \xi(x), \eta(y) \rangle$  for every  $x, y \in G$ . The Herz–Schur norm of  $\varphi$  is defined by

$$\|\varphi\|_{\mathrm{cb}} = \inf\{\|\xi\|_\infty\|\eta\|_\infty\},$$

where the infimum is taken over all  $\xi, \eta \in C(G, \mathcal{H})$  as above. The Banach space of Herz–Schur multipliers is denoted by  $B_2(G)$ . Clearly, one has a contractive embedding of  $\mathcal{A}G$  into  $B_2(G)$ . The Herz–Schur norm  $\|\varphi\|_{\mathrm{cb}}$  coincides with the cb-norm of the corresponding multipliers on  $\mathcal{L}G$  or on  $C_\lambda^*G$ :

$$\|\varphi\|_{\mathrm{cb}} = \|m_\varphi: \mathcal{L}G \ni \lambda(f) \mapsto \lambda(\varphi f) \in \mathcal{L}G\|_{\mathrm{cb}} = \|m_\varphi|_{C_\lambda^*G}\|_{\mathrm{cb}}.$$

Indeed,  $\|\varphi\|_{\mathrm{cb}} \geq \|m_\varphi\|_{\mathrm{cb}}$  is easy to see: Given a factorization  $\varphi(x^{-1}y) = \langle \xi(x), \eta(y) \rangle$  with  $\xi, \eta \in C(G, \mathcal{H})$ , we define  $V_\xi: L^2(G) \rightarrow L^2(G, \mathcal{H})$  by  $(V_\xi f)(x) = f(x)\xi(x^{-1})$ , and likewise for  $V_\eta$ . Then,  $\lambda(\varphi f) = V_\eta^*(\lambda(f) \otimes 1_{\mathcal{H}})V_\xi$  and  $\|m_\varphi\|_{\mathrm{cb}} \leq \|\xi\|_\infty\|\eta\|_\infty$ . We will give a proof of the converse inequality in Lemma 1, but sketch it here in the case of amenable groups. Let  $N$  be an amenable group and  $\varphi \in B_2(N)$ . Since the unit character  $\tau_0$  is continuous on  $C_\lambda^*N$ , the linear functional  $\omega_\varphi = \tau_0 \circ m_\varphi$  is bounded on  $C_\lambda^*N$  and satisfies  $\|\omega_\varphi\| \leq \|m_\varphi\|_{\mathrm{cb}}$ . Let  $(\pi, \mathcal{H})$  be the GNS representation for  $|\omega_\varphi|$  and view  $\pi$  as a continuous unitary  $N$ -representation. Then, there are vectors  $\xi, \eta \in \mathcal{H}$  such that  $\|\xi\|_\infty\|\eta\|_\infty = \|\omega_\varphi\|$  and  $\varphi(x) = \langle \pi(x)\xi, \eta \rangle$  for every  $x \in N$ . (Hence,  $\|\omega_\varphi\| = \|\varphi\|_{\mathrm{cb}}$ .)

**Definition.** Let  $G$  be a group. By an *approximate identity* on  $G$ , we mean a net  $(\varphi_n)$  in  $\mathcal{A}G$  which converges to 1 uniformly on compacta. It is *completely bounded* if

$$\|(\varphi_n)\|_{\text{cb}} := \sup_n \|\varphi_n\|_{\text{cb}} < +\infty.$$

A group  $G$  is said to be *weakly amenable* if there is a completely bounded approximate identity on  $G$ . The Cowling–Haagerup constant  $\Lambda_{\text{cb}}(G)$  is defined to be

$$\Lambda_{\text{cb}}(G) = \inf \{ \|(\varphi_n)\|_{\text{cb}} : (\varphi_n) \text{ a c.b.a.i. on } G \}.$$

Note that the above infimum is attained. See [CH, BO] for more information.

It is easy to see that if  $H \leq G$  is a closed subgroup, then  $\Lambda_{\text{cb}}(H) \leq \Lambda_{\text{cb}}(G)$ . On this occasion, we record that the same inequality holds also for a “random” or “ME” subgroup in the sense of [Mo, Sa] (cf. [CZ]). For this, we only consider countable discrete groups  $\Lambda$  and  $\Gamma$ . Recall that  $\Lambda$  is an ME subgroup of  $\Gamma$  if there is a standard measure space  $\Omega$  on which  $\Lambda \times \Gamma$  acts by measure-preserving transformations in such a way that each of  $\Lambda$ - and  $\Gamma$ -actions admits a fundamental domain and the measure of  $\Omega_\Gamma := \Omega/\Gamma$  is finite. The action  $\Lambda \curvearrowright \Omega$  gives rise to a measure-preserving action  $\Lambda \curvearrowright \Omega_\Gamma$  and a measurable cocycle  $\alpha: \Lambda \times \Omega_\Gamma \rightarrow \Gamma$  such that the action  $\Lambda \curvearrowright \Omega$  is isomorphic (up to null sets) to the twisted action  $\Lambda \curvearrowright \Omega_\Gamma \times \Gamma$ , given by  $a(t, g) = (at, \alpha(a, t)g)$  for  $a \in \Lambda$ ,  $t \in \Omega_\Gamma$  and  $g \in \Gamma$ . The map  $\alpha$  satisfies the cocycle identity:  $\alpha(ab, t) = \alpha(a, bt)\alpha(b, t)$  for every  $a, b \in \Lambda$  and a.e.  $t \in \Omega_\Gamma$ . For  $\varphi \in B_2(\Gamma)$ , we denote the “induced” function on  $\Lambda$  by  $\varphi_\alpha$ :

$$\varphi_\alpha(a) = \int_{\Omega_\Gamma} \varphi(\alpha(a, t)) dt.$$

Here, we normalized the measure so that  $|\Omega_\Gamma| = 1$ . Since

$$\varphi_\alpha(b^{-1}a) = \int_{\Omega_\Gamma} \varphi(\alpha(b, b^{-1}at)^{-1}\alpha(a, t)) dt = \int_{\Omega_\Gamma} \varphi(\alpha(b, b^{-1}t)^{-1}\alpha(a, a^{-1}t)) dt,$$

one has  $\varphi_\alpha \in B_2(\Lambda)$  and  $\|\varphi_\alpha\|_{\text{cb}} \leq \|\varphi\|_{\text{cb}}$ . Suppose now that  $\varphi \in \mathcal{A}\Gamma$ . Then,  $\varphi_\alpha$  is a coefficient of the unitary  $\Lambda$ -representation  $\sigma$  on  $L^2(\Omega)$  induced by the measure-preserving action  $\Lambda \curvearrowright \Omega$ , i.e., there are  $\xi, \eta \in L^2(\Omega)$  such that  $\varphi_\alpha(a) = \langle \sigma(a)\xi, \eta \rangle$ . Since  $\Omega$  admits a  $\Lambda$ -fundamental domain,  $\sigma$  is a multiple of the regular representation and  $\varphi_\alpha \in \mathcal{A}\Lambda$ . By inducing an approximate identity on  $\Gamma$ , one sees that if  $\Gamma$  is weakly amenable, then so is  $\Lambda$  and  $\Lambda_{\text{cb}}(\Lambda) \leq \Lambda_{\text{cb}}(\Gamma)$ .

### 3. PROOF OF THEOREM A

**Lemma 1.** *Let  $N$  be an amenable closed normal subgroup of  $G$  and  $\varphi \in B_2(G)$ . Then, there are a Hilbert space  $\mathcal{H}$ , functions  $\xi, \eta \in C(G, \mathcal{H})$  and a continuous unitary representation  $\pi$  of  $N$  on  $\mathcal{H}$  such that*

- $\|\xi\|_\infty = \|\eta\|_\infty = \|\varphi\|_{\text{cb}}^{1/2}$ ;
- $\varphi(y^{-1}x) = \langle \xi(x), \eta(y) \rangle$  for every  $x, y \in G$ ;
- $\pi(a)\xi(x) = \xi(ax)$  and  $\pi(a)\eta(y) = \eta(ay)$  for every  $a \in N$  and  $x, y \in G$ .

*Proof.* We follow Jolissaint's simple proof ([Jo]) of the inequality  $\|\varphi\|_{\text{cb}} \leq \|m_\varphi\|_{\text{cb}}$ . Since  $N$  is amenable, the quotient map  $q: G \rightarrow G/N$  extends to a  $*$ -homomorphism  $q: C_\lambda^*G \rightarrow C_\lambda^*(G/N)$  between the reduced group  $C^*$ -algebras. Since  $q \circ m_\varphi$  is completely bounded on  $C_\lambda^*G$ , a Stinespring type factorization theorem (Theorem B.7 in [BO]) yields a  $*$ -representation  $\pi: C_\lambda^*G \rightarrow \mathbb{B}(\mathcal{H})$  and operators  $V, W \in \mathbb{B}(L^2(G/N), \mathcal{H})$  such that  $\|V\| = \|W\| \leq \|q \circ m_\varphi\|_{\text{cb}}^{1/2}$  and  $(q \circ m_\varphi)(X) = W^*\pi(X)V$  for  $X \in C_\lambda^*G$ . We view  $\pi$  as a continuous unitary representation of  $G$ . Then, for a fixed unit vector  $\zeta \in L^2(G/N)$ , the maps  $\xi(x) = \pi(x)V\lambda_{G/N}(q(x^{-1}))\zeta$  and  $\eta(y) = \pi(y)W\lambda_{G/N}(q(y^{-1}))\zeta$  are continuous,  $\|\xi\|_\infty, \|\eta\|_\infty \leq \|m_\varphi\|_{\text{cb}}^{1/2}$  and  $\varphi(y^{-1}x) = \langle \xi(x), \eta(y) \rangle$  for every  $x, y \in G$ . Moreover,  $\pi(a)\xi(x) = \xi(ax)$  for  $a \in N$ , because  $\lambda_{G/N}(a) = 1$ .  $\square$

We denote by  $\varphi^g$  the right translation of a function  $\varphi$  by  $g \in G$ , i.e.,  $\varphi^g(x) = \varphi(xg^{-1})$ .

**Lemma 2.** *Let  $N$  be an amenable group,  $\varphi \in B_2(N)$  and  $a \in N$ . Then,*

$$\left\| \frac{1}{2}(\varphi + \varphi^a) \right\|_{\text{cb}}^2 + \left\| \frac{1}{2}(\varphi - \varphi^a) \right\|_{\text{cb}}^2 \leq \|\varphi\|_{\text{cb}}^2.$$

*Proof.* There are a continuous unitary representation  $\pi$  of  $N$  on a Hilbert space  $\mathcal{H}$  and vectors  $\xi, \eta \in \mathcal{H}$  such that  $\|\xi\| = \|\eta\| = \|\varphi\|_{\text{cb}}^{1/2}$  and  $\varphi(x) = \langle \pi(x)\xi, \eta \rangle$  for every  $x \in N$ . Since  $(\varphi \pm \varphi^a)(x) = \langle \pi(x)(\xi \pm \pi(a^{-1})\xi), \eta \rangle$ , one has

$$\|\varphi + \varphi^a\|_{\text{cb}}^2 + \|\varphi - \varphi^a\|_{\text{cb}}^2 \leq \|\xi + \pi(a^{-1})\xi\|^2 \|\eta\|^2 + \|\xi - \pi(a^{-1})\xi\|^2 \|\eta\|^2 = 4\|\varphi\|_{\text{cb}}^2. \quad \square$$

For  $\varphi \in B_2(G)$ , we define  $\varphi^*(x) := \overline{\varphi(x^{-1})}$ , and say  $\varphi$  is *self-adjoint* if  $\varphi^* = \varphi$ . For any  $\varphi \in B_2(G)$ , the function  $(\varphi + \varphi^*)/2$  is self-adjoint and  $\|(\varphi + \varphi^*)/2\|_{\text{cb}} \leq \|\varphi\|_{\text{cb}}$ . Thus every approximate identity can be made self-adjoint without increasing norm. We fix a closed subgroup  $N$  of  $G$ . A completely bounded approximate identity  $(\varphi_n)$  on  $G$  is said to be  *$N$ -optimal* if all  $\varphi_n$  are self-adjoint,  $\|(\varphi_n)\|_{\text{cb}} = \Lambda_{\text{cb}}(G)$  and

$$\|(\varphi_n|_N)\|_{\text{cb}} = \inf\{\|(\psi_n|_N)\|_{\text{cb}} : (\psi_n) \text{ a c.b.a.i. such that } \|(\psi_n)\|_{\text{cb}} = \Lambda_{\text{cb}}(G)\}.$$

Note that an  $N$ -optimal approximate identity exists (if  $G$  is weakly amenable).

**Proposition 3.** *Let  $G$  be an weakly amenable group and  $N$  be an amenable closed normal subgroup of  $G$ . Let  $(\varphi_n)$  be an  $N$ -optimal approximate identity on  $G$ . Then, for every  $g \in G$  and  $a \in N$ ,*

$$\lim_n \|(\varphi_n - \varphi_n \circ \text{Ad}_g)|_N\|_{\text{cb}} = 0 \text{ and } \lim_n \|(\varphi_n - \varphi_n^a)|_N\|_{\text{cb}} = 0.$$

*Proof.* We apply Lemma 1 for each  $\varphi_n$  and find  $(\pi_n, \mathcal{H}_n, \xi_n, \eta_n)$  satisfying the conditions stated there. In particular,  $\|\xi\|_\infty = \|\eta\|_\infty \leq \Lambda_{\text{cb}}(G)^{1/2}$  and  $\varphi_n(y^{-1}x) = \langle \xi_n(x), \eta_n(y) \rangle$  for every  $x, y \in G$ . Let  $g \in G$  be given and consider  $\psi_n = (\varphi_n + \varphi_n^g)/2$ . Since  $(\psi_n)$  is a completely bounded approximate identity, one must have  $\liminf_n \|\psi_n\|_{\text{cb}} \geq \Lambda_{\text{cb}}(G)$ . Meanwhile, since  $\varphi_n$  is self-adjoint,

$$\psi_n(y^{-1}x) = \frac{1}{4}(\langle \xi_n(x) + \xi_n(xg^{-1}), \eta_n(y) \rangle + \langle \eta_n(x) + \eta_n(xg^{-1}), \xi_n(y) \rangle)$$

and hence

$$\|\psi_n\|_{\text{cb}} \leq \left\| \frac{1}{\sqrt{2}} \left( \frac{\xi_n + \xi_n^g}{2}, \frac{\eta_n + \eta_n^g}{2} \right) \right\|_{L^\infty(G, \mathcal{H} \oplus \mathcal{H})} \left\| \frac{1}{\sqrt{2}} (\eta_n, \xi_n) \right\|_{L^\infty(G, \mathcal{H} \oplus \mathcal{H})} \leq \Lambda_{\text{cb}}(G).$$

It follows that

$$\lim_n \left\| \frac{1}{\sqrt{2}} \left( \frac{\xi_n + \xi_n^g}{2}, \frac{\eta_n + \eta_n^g}{2} \right) \right\|_{L^\infty(G, \mathcal{H} \oplus \mathcal{H})} = \Lambda_{\text{cb}}(G)^{1/2},$$

which means that there is a net  $z_n \in G$  such that

$$\lim_n \left\| \frac{\xi_n(z_n) + \xi_n(z_n g^{-1})}{2} \right\| = \Lambda_{\text{cb}}(G)^{1/2} \text{ and } \lim_n \left\| \frac{\eta_n(z_n) + \eta_n(z_n g^{-1})}{2} \right\| = \Lambda_{\text{cb}}(G)^{1/2}.$$

By the parallelogram identity, this implies that

$$\lim_n \|\xi_n(z_n) - \xi_n(z_n g^{-1})\| = 0 \text{ and } \lim_n \|\eta_n(z_n) - \eta_n(z_n g^{-1})\| = 0.$$

The unitary  $N$ -representation  $\pi'_n = \pi_n \circ \text{Ad}_{z_n}$  satisfies  $\pi'_n(a)\xi_n(x) = \xi_n(z_n a z_n^{-1}x)$ ,

$$\varphi_n(a) = \langle \pi'_n(a)\xi_n(z_n), \eta_n(z_n) \rangle \text{ and } (\varphi_n \circ \text{Ad}_g)(a) = \langle \pi'_n(a)\xi_n(z_n g^{-1}), \eta_n(z_n g^{-1}) \rangle$$

for  $a \in N$ . It follows that  $\|(\varphi_n - \varphi_n \circ \text{Ad}_g)|_N\|_{\text{cb}} \rightarrow 0$ . That  $\|(\varphi_n - \varphi_n^a)|_N\|_{\text{cb}} \rightarrow 0$  follows from  $N$ -optimality of  $(\varphi_n)$  and Lemma 2.  $\square$

*Proof of Theorem A.* Let  $(\varphi_n)$  be an  $N$ -optimal approximate identity on  $G$  and consider linear functionals  $\omega_n = \tau_0 \circ m_{\varphi_n}$  on  $C_\lambda^*N$ , where  $\tau_0$  is the unit character on  $N$  (see Section 2). Since  $\varphi_n \in \mathcal{AG}$ , the linear functionals  $\omega_n$  extend to ultraweakly-continuous linear functionals on the group von Neumann algebra  $\mathcal{LN}$ . Indeed, they are nothing but  $\varphi_n|_N \in \mathcal{AN} = (\mathcal{LN})_*$ . One has  $\|\omega_n\| \leq \Lambda_{\text{cb}}(G)$ ,  $\omega_n(1_{\mathcal{LN}}) = \varphi_n(1_N)$  and, by Proposition 3,  $\|\omega_n - \omega_n \circ \text{Ad}_g\| \rightarrow 0$  and  $\|\omega_n - \omega_n^a\| \rightarrow 0$  for every  $g \in G$  and  $a \in N$ . We consider  $\zeta_n := |\omega_n|^{1/2} \in L^2(N)$  and  $\zeta'_n := \omega_n |\omega_n|^{-1/2} \in L^2(N)$  so that  $\omega_n(X) = \langle X\zeta_n, \zeta'_n \rangle$  for  $X \in \mathcal{LN}$ . Here the absolute value and the square root are taken in the sense of the standard representation  $\mathcal{LN} \subset \mathbb{B}(L^2(N))$ . (In case where  $N$  is abelian, the Fourier transform  $L^2(N) \cong L^2(\hat{N})$  implements  $\mathcal{LN} \cong L^\infty(\hat{N})$  and  $(\mathcal{LN})_* \cong L^1(\hat{N})$ , and the absolute value and square root are computed as ordinary functions on the Pontrjagin dual  $\hat{N}$ .) We note that  $\varphi_n(1) \leq \|\zeta_n\|_2^2 \leq \Lambda_{\text{cb}}(G)$ . By continuity of the absolute value (Proposition III.4.10 in [Ta]) and the Powers–Størmer inequality, one has  $\|\zeta_n - \text{Ad}_g \zeta_n\|_2 \rightarrow 0$  for every  $g \in G$ . Moreover, since

$$\|\zeta_n\|_2 \|\zeta'_n\|_2 - \left\| \frac{\zeta_n + \lambda(a^{-1})\zeta_n}{2} \right\|_2 \|\zeta'_n\|_2 \leq \|\omega_n\| - \left\| \frac{\omega_n + \omega_n^a}{2} \right\| \rightarrow 0,$$

one has  $\|\zeta_n - \lambda(a^{-1})\zeta_n\|_2 \rightarrow 0$  for every  $a \in N$ . Thus, any limit point of  $(\zeta_n^2)$  in  $L^\infty(N)^*$  is a non-zero positive  $G \ltimes N$ -invariant linear functional on  $L^\infty(N)$ .  $\square$

**Corollary 4.** *Let  $\Gamma$  and  $\Lambda$  be discrete groups with  $\Lambda$  non-trivial and  $\Gamma$  non-amenable. Then the wreath product  $\Lambda \wr \Gamma$  is not weakly amenable. Also, the group  $\text{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$  is not weakly amenable.*

*Proof.* The proof is same as that of Corollary 2.12 in [OP]. We note that the stabilizer of a non-neutral element in  $\mathbb{Z}^2$  is an abelian (amenable) subgroup of  $\mathrm{SL}(2, \mathbb{Z})$ .  $\square$

#### 4. PROOF OF THEOREM B

We first fix notations. Throughout this section,  $M$  is a finite von Neumann algebra with a distinguished faithful normal tracial state  $\tau$ , and  $P$  is an amenable von Neumann subalgebra of  $M$ . The *normalizer*  $\mathcal{N}(P)$  of  $P$  in  $M$  is

$$\mathcal{N}(P) = \{u \in \mathcal{U}(M) : \mathrm{Ad}_u(P) = P\},$$

where  $\mathcal{U}(M)$  is the group of the unitary elements of  $M$  and  $\mathrm{Ad}_u(x) = uxu^*$ . The GNS Hilbert space with respect to the trace  $\tau$  is denoted by  $L^2(M)$  and the vector in  $L^2(M)$  associated with  $x \in M$  is denoted by  $\hat{x}$ , i.e.,  $\langle \hat{x}, \hat{y} \rangle = \tau(y^*x)$  for  $x, y \in M$ . The complex conjugate  $\bar{M} = \{\bar{a} : a \in M\}$  of  $M$  acts on  $L^2(M)$  from the right. Thus there is a  $*$ -representation  $\varsigma$  of the algebraic tensor product  $M \otimes \bar{M}$  on  $L^2(M)$  defined by  $\varsigma(a \otimes \bar{b})\hat{x} = \widehat{axb^*}$  for  $a, b, x \in M$ . We also use the bimodule notation  $a\hat{x}b^*$  for  $\varsigma(a \otimes \bar{b})\hat{x}$ . Since  $P$  is amenable, the  $*$ -homomorphism  $\varsigma|_{M \otimes \bar{P}}$  is continuous with respect to the minimal tensor norm.

**Definition.** A von Neumann algebra  $M$  is said to have the *weak\* completely bounded approximation property*, or  $W^*\mathrm{CBAP}$  in short, if there is a net of ultraweakly-continuous finite-rank maps  $(\varphi_n)$  on  $M$  such that  $\varphi_n \rightarrow \mathrm{id}_M$  in the point-ultraweak topology and  $\sup \|\varphi_n\|_{\mathrm{cb}} < +\infty$ .

Recall that a finite von Neumann algebra  $P$  is amenable (a.k.a. hyperfinite, injective, AFD, etc.) if the trace  $\tau$  on  $P$  extends to a  $P$ -central state  $\omega$  on  $\mathbb{B}(L^2(P))$ . Here, a state  $\omega$  is said to be  *$P$ -central* if  $\omega \circ \mathrm{Ad}_u = \omega$  for every  $u \in \mathcal{U}(P)$ , or equivalently  $\omega(ax) = \omega(xa)$  for every  $a \in P$  and  $x \in \mathbb{B}(L^2(P))$ .

**Definition.** Let  $P$  be a finite von Neumann algebra and  $\mathcal{G}$  be a group acting on  $P$  by trace-preserving  $*$ -automorphisms. We denote by  $\sigma$  the corresponding unitary representation of  $\mathcal{G}$  on  $L^2(P)$ . The action  $\mathcal{G} \curvearrowright P$  is said to be *weakly compact* if there is a state  $\omega$  on  $\mathbb{B}(L^2(P))$  such that  $\omega|_P = \tau$  and  $\omega \circ \mathrm{Ad}_u = \omega$  for every  $u \in \sigma(\mathcal{G}) \cup \mathcal{U}(P)$ . (This forces  $P$  to be amenable.) A von Neumann subalgebra  $P$  of a finite von Neumann algebra  $M$  is said to be *weakly compact* in  $M$  if the conjugate action by the normalizer  $\mathcal{N}(P)$  is weakly compact. See [OP] for more information.

If  $M$  admits a crossed product decomposition  $M = P \rtimes \Lambda$  such that the “core”  $P$  is non-atomic and weakly compact in  $M$ , then  $M$  does not have property (T). Indeed, the hypothesis implies that  $\mathcal{L}\Lambda$  is co-amenable in  $M$  (Proposition 3.2 in [OP]), i.e., the  $M$ - $M$  module  $L^2\langle M, e_{\mathcal{L}\Lambda} \rangle$  contains an approximately central vector (see Theorem 2.1 in [OP]). But since  $L^2\langle M, e_{\mathcal{L}\Lambda} \rangle \cong \bigoplus_{t \in \Lambda} L^2(P) \otimes L^2(P)$  as a  $P$ - $P$  module, it does not contain a non-zero central vector. This proves non property (T) of  $M$ .

**Lemma 5.** *Every  $P$ -central state  $\omega$  on  $\mathbb{B}(L^2(P))$  decomposes uniquely as a sum  $\omega = \omega_n + \omega_s$  of  $P$ -central positive linear functionals such that  $\omega_n|_P$  is normal and  $\omega_s|_P$  is singular. A trace-preserving action  $\mathcal{G} \curvearrowright P$  is weakly compact if there is a positive linear functional  $\omega$  on  $\mathbb{B}(L^2(P))$  such that*

- $\omega(p) > 0$  for every non-zero central projection  $p$  in  $P$ ,
- $\omega \circ \text{Ad}_u = \omega$  for every  $u \in \sigma(\mathcal{G}) \cup \mathcal{U}(P)$ .

*Proof.* We denote by  $Z$  the center of  $P$ . Recall that every tracial state  $\tau'$  on  $P$  satisfies  $\tau' = \tau'|_Z \circ E_Z$ , where  $E_Z: P \rightarrow Z$  is the center-valued trace. In particular,  $\tau'$  is normal on  $P$  if and only if it is normal on  $Z$ . Let  $\omega$  be a  $P$ -central state and consider the normal/singular decomposition of the state  $\omega|_Z$  (see Definition III.2.15 in [Ta]). There is an increasing sequence  $(p_n)$  of projections in  $Z$  such that  $p_n \nearrow 1$  and  $(\omega|_Z)_s(p_n) = 0$  for all  $n$  (see Theorem III.3.8 in [Ta]). We fix an ultralimit  $\text{Lim}$  on  $\mathbb{N}$  and let  $\omega_n(x) = \text{Lim } \omega(p_n x)$  and  $\omega_s = \omega - \omega_n$ . Since  $\omega$  is  $P$ -central, these are  $P$ -central positive linear functionals on  $\mathbb{B}(L^2(P))$ , and  $\omega|_Z = \omega_n|_Z + \omega_s|_Z$  is the normal/singular decomposition of  $\omega|_Z$ . Suppose that  $\omega = \omega'_n + \omega'_s$  is another such decomposition. Then, since  $\omega_s + \omega'_s$  is singular on  $Z$ , there is an increasing sequence  $(q_n)$  of projections in  $Z$  such that  $q_n \nearrow 1$  and  $(\omega_s + \omega'_s)(q_n) = 0$  for all  $n$ . It follows that  $\omega'_n(x) = \lim \omega(q_n x) = \omega_n(x)$  for every  $x \in \mathbb{B}(L^2(P))$ . This proves the first half of this lemma. For the second half, we first observe that we may assume  $\omega$  is normal on  $P$  by uniqueness of the normal/singular decomposition. Thus, there is  $h \in L^1(Z)_+$  such that  $\omega(z) = \tau(hz)$  for  $z \in Z$ . By assumption,  $h$  has full support and is  $\mathcal{G}$ -invariant. Thus  $\tilde{\omega}(x) := \text{Lim } \omega((h + n^{-1})^{-1}x)$  defines a  $\mathcal{G}$ -invariant  $P$ -central state on  $\mathbb{B}(L^2(P))$  such that  $\tilde{\tau}|_Z = \tau|_Z$ .  $\square$

**Lemma 6.** *Let  $\varphi$  be a completely bounded map on  $M$ . Then, there are a  $*$ -representation of the minimal tensor product  $M \otimes_{\min} \bar{P}$  on a Hilbert space  $\mathcal{H}$  and operators  $V, W \in \mathbb{B}(L^2(M), \mathcal{H})$  such that  $\|V\| = \|W\| \leq \|\varphi\|_{\text{cb}}^{1/2}$  and*

$$\tau(y^* \varphi(a) x b^*) = \langle \varphi(a) \hat{x} b^*, \hat{y} \rangle = \langle \pi(a \otimes \bar{b}) V \hat{x}, W \hat{y} \rangle$$

for every  $a, x, y \in M$  and  $b \in P$ .

*Proof.* Since the  $*$ -representation  $\varsigma: M \otimes_{\min} \bar{P} \rightarrow \mathbb{B}(L^2(M))$  is continuous, a Stinespring type factorization theorem (Theorem B.7 in [BO]), applied to the completely bounded map  $\varsigma \circ (\varphi \otimes \text{id}_{\bar{P}})$ , yields a  $*$ -representation  $\pi: M \otimes_{\min} \bar{P} \rightarrow \mathbb{B}(\mathcal{H})$  and operators  $V, W \in \mathbb{B}(L^2(M), \mathcal{H})$  such that  $\|V\| \|W\| \leq \|\varphi\|_{\text{cb}}$  and

$$\varphi(a) \hat{x} b^* = \varsigma((\varphi \otimes \text{id}_{\bar{P}})(a \otimes \bar{b})) \hat{x} = W^* \pi(a \otimes \bar{b}) V \hat{x}$$

for  $a, x \in M$  and  $b \in P$ .  $\square$

Since  $W^*\text{CBAP}$  passes to a subalgebra (which is the range of a conditional expectation), we assume from now on that  $P$  is *regular* in  $M$ , i.e.,  $\mathcal{N}(P)$  generates  $M$  as a von Neumann algebra. We say a linear map  $\varphi$  on  $M$  is  $P$ -cb if there are a  $*$ -representation  $\pi$  of  $M \otimes_{\min} \bar{P}$  on a Hilbert space  $\mathcal{H}$  and functions  $V, W \in \ell_\infty(\mathcal{N}(P), \mathcal{H})$  such that

$$(*) \quad \langle \varphi(a) \hat{x} b^*, \hat{y} \rangle = \langle \pi(a \otimes \bar{b}) V(x), W(y) \rangle$$

for every  $a \in M$ ,  $x, y \in \mathcal{N}(P)$  and  $b \in P$ . The  $P$ -cb norm of  $\varphi$  is defined as

$$\|\varphi\|_P = \inf\{\|V\|_\infty\|W\|_\infty : (\pi, \mathcal{H}, V, W) \text{ satisfies } (*)\}.$$

It is indeed a norm and the infimum is attained (for the latter fact, use ultraproduct). By the above lemma,  $\|\varphi\|_P \leq \|\varphi\|_{cb}$ . By an *approximate identity*, we mean a net  $(\varphi_n)$  of ultraweakly-continuous finite-rank maps such that  $\varphi_n \rightarrow \text{id}_M$  in the point-ultraweak topology and  $\sup \|\varphi_n\|_P < +\infty$ . It exists if  $M$  has the  $W^*$ CBAP. We define

$$\Lambda_P(M) = \inf\{\sup_n \|\varphi_n\|_P : (\varphi_n) \text{ an approximate identity}\}.$$

For a map  $\varphi$  on  $M$ , we define  $\varphi^*(a) = \varphi(a^*)^*$  and say  $\varphi$  is *self-adjoint* if  $\varphi = \varphi^*$ . We note that if  $(\pi, \mathcal{H}, V, W)$  satisfies  $(*)$  for  $\varphi$ , then  $(\pi, \mathcal{H}, W, V)$  satisfies  $(*)$  for  $\varphi^*$ . In particular,  $(\varphi + \varphi^*)/2$  is self-adjoint and  $\|(\varphi + \varphi^*)/2\|_P \leq \|\varphi\|_P$ . Thus, any approximate identity can be made self-adjoint without increasing norm. For a  $P$ -cb map  $\varphi$ , we define a bounded linear functional  $\mu_\varphi$  on  $M \otimes_{\min} \bar{P}$  by

$$\mu_\varphi(a \otimes \bar{b}) := \tau(\varphi(a)b^*) = \langle \varphi(a)\hat{1}b^*, \hat{1} \rangle = \langle \pi(a \otimes \bar{b})V(1), W(1) \rangle.$$

Note that  $\|\mu_\varphi\| \leq \|\varphi\|_P$ . If  $\varphi$  is ultraweakly-continuous and finite-rank, then  $\mu_\varphi$  extend to an ultraweakly-continuous linear functional on the von Neumann algebra  $M \bar{\otimes} \bar{P}$ .

**Proposition 7.** *Let  $M$  be a finite von Neumann algebra having the  $W^*$ CBAP and  $(\varphi_n)$  be a self-adjoint approximate identity such that  $\sup_n \|\varphi_n\|_P = \Lambda_P(M)$ . Then, the net  $\mu_n := \mu_{\varphi_n}|_{P \bar{\otimes} \bar{P}}$  satisfies the following properties:*

- $\mu_n$  are self-adjoint and ultraweakly-continuous for all  $n$ ;
- $\sup \|\mu_n\| \leq \Lambda_P(M)$  and  $\mu_n(a \otimes \bar{1}) \rightarrow \tau(a)$  for every  $a \in P$ ;
- $\|\mu_n - \mu_n^{v \otimes \bar{v}}\| \rightarrow 0$  for every  $v \in \mathcal{U}(P)$ , where  $\mu_n^{v \otimes \bar{v}}(a \otimes \bar{b}) = \mu_n((a \otimes \bar{b})(v \otimes \bar{v})^*)$ ;
- $\|\mu_n - \mu_n \circ \text{Ad}_{u \otimes \bar{u}}\| \rightarrow 0$  for every  $u \in \mathcal{N}(P)$ .

*Proof.* The first two conditions are easy to see. Let  $u \in \mathcal{N}(P)$  be given, and define  $\varphi_n^u$  by  $\varphi_n^u(a) = \varphi_n(au^*)u$  for  $a \in M$ . We note that  $\mu_{\varphi_n^u}|_{P \bar{\otimes} \bar{P}} = \mu_n^{u \otimes \bar{u}}$  if  $u \in \mathcal{U}(P)$ . Thus, it suffices to show

$$\lim_n \|\mu_{\varphi_n} - \mu_{\varphi_n^u}\| = 0 \text{ and } \lim_n \|\mu_{\varphi_n} - \mu_{\varphi_n} \circ \text{Ad}_{u \otimes \bar{u}}\| = 0.$$

Take  $(\pi_n, \mathcal{H}_n, V_n, W_n)$  satisfying  $(*)$  and  $\lim \|V_n\|_\infty = \lim \|W_n\|_\infty = \Lambda_P(M)^{1/2}$ . It follows that

$$\langle \varphi_n^u(a)\hat{x}b^*, \hat{y} \rangle = \langle \varphi_n(au^*)\widehat{u}xb^*, \hat{y} \rangle = \langle \pi_n(a \otimes \bar{b})\pi_n(u^* \otimes \bar{1})V_n(ux), W_n(y) \rangle$$

for every  $a \in M$ ,  $b \in P$  and  $x, y \in \mathcal{N}(P)$ . Hence with  $V_n^u(x) = \pi_n(u^* \otimes \bar{1})V_n(ux)$ , the quadruplet  $(\pi_n, \mathcal{H}_n, V_n^u, W_n)$  satisfies  $(*)$  for  $\varphi_n^u$ . Note that  $\|V_n^u\|_\infty = \|V_n\|_\infty$ . We define  $W_n^u$  similarly. Since  $\varphi_n$  is self-adjoint,  $(\pi_n, \mathcal{H}_n, W_n, V_n)$  (resp.  $(\pi_n, \mathcal{H}_n, W_n^u, V_n)$ ) satisfies  $(*)$  for  $\varphi_n$  (resp.  $\varphi_n^u$ ), too. Thus, for  $\psi_n = (\varphi_n + \varphi_n^u)/2$ , one has

$$\|\psi_n\|_P \leq \left\| \frac{1}{\sqrt{2}} \left( \frac{V_n + V_n^u}{2}, \frac{W_n + W_n^u}{2} \right) \right\|_{\ell_\infty(\mathcal{N}(P), \mathcal{H} \oplus \mathcal{H})} \left\| \frac{1}{\sqrt{2}} (W_n, V_n) \right\|_{\ell_\infty(\mathcal{N}(P), \mathcal{H} \oplus \mathcal{H})}.$$



Meanwhile, since  $(\psi_n)$  is an approximate identity, one must have  $\liminf \|\psi_n\|_P \geq \Lambda_P(M)$ . It follows that

$$\lim_n \left\| \frac{1}{\sqrt{2}} \left( \frac{V_n + V_n^u}{2}, \frac{W_n + W_n^u}{2} \right) \right\|_{\ell_\infty(\mathcal{N}(P), \mathcal{H} \oplus \mathcal{H})} = \Lambda_P(M)^{1/2}$$

and hence there is a net  $(z_n)$  in  $\mathcal{N}(P)$  such that

$$\lim_n \left\| \frac{1}{\sqrt{2}} \left( \frac{(V_n + V_n^u)(z_n)}{2}, \frac{(W_n + W_n^u)(z_n)}{2} \right) \right\|_{\mathcal{H} \oplus \mathcal{H}} = \Lambda_P(M)^{1/2}.$$

By the parallelogram identity, this implies that

$$\lim_n \|V_n(z_n) - V_n^u(z_n)\| = 0 \text{ and } \lim_n \|W_n(z_n) - W_n^u(z_n)\| = 0.$$

Let  $\pi'_n = \pi_n \circ (\text{id}_M \otimes \text{Ad}_{\bar{z}_n^{-1}})$ . Since

$$\begin{aligned} \mu_{\varphi_n}(a \otimes \bar{b}) &= \langle \varphi_n(a) \hat{z}_n \text{Ad}_{z_n^{-1}}(b)^*, \hat{z}_n \rangle = \langle \pi'_n(a \otimes \bar{b}) V_n(z_n), W_n(z_n) \rangle, \\ \mu_{\varphi_n^u}(a \otimes \bar{b}) &= \langle \varphi_n(au^*) \widehat{u z_n} \text{Ad}_{z_n^{-1}}(b)^*, \hat{z}_n \rangle = \langle \pi'_n(a \otimes \bar{b}) V_n^u(z_n), W_n(z_n) \rangle, \end{aligned}$$

and

$$(\mu_{\varphi_n} \circ \text{Ad}_{u \otimes \bar{u}})(a \otimes \bar{b}) = \langle \varphi_n(uau^*) \widehat{u z_n} \text{Ad}_{z_n^{-1}}(b)^*, \widehat{u z_n} \rangle = \langle \pi'_n(a \otimes \bar{b}) V_n^u(z_n), W_n^u(z_n) \rangle,$$

we conclude that  $\|\mu_{\varphi_n} - \mu_{\varphi_n^u}\| \rightarrow 0$  and  $\|\mu_{\varphi_n} - \mu_{\varphi_n} \circ \text{Ad}_{u \otimes \bar{u}}\| \rightarrow 0$ .  $\square$

*Proof of Theorem B.* Since  $M$  has the  $W^*$ CBAP, there is a net  $(\mu_n)$  satisfying the conclusion of Proposition 7. We view  $\mu_n$  as an element in  $L^1(P \bar{\otimes} \bar{P})$  (see Section 2 in [OP]) and let  $\zeta_n = |\mu_n|^{1/2} \in L^2(P \bar{\otimes} \bar{P})$  and  $\zeta'_n = \mu_n |\mu_n|^{-1/2} \in L^2(P \bar{\otimes} \bar{P})$  so that  $\mu_n(X) = \langle X \zeta_n, \zeta'_n \rangle$  for  $X \in P \bar{\otimes} \bar{P}$ . By continuity of the absolute value (Proposition III.4.10 in [Ta]) and the Powers–Størmer inequality, one has  $\|\zeta_n - \text{Ad}_{u \otimes \bar{u}} \zeta_n\|_2 \rightarrow 0$  for every  $u \in \mathcal{N}(P)$ . Since

$$2\|\mu_n\| \approx \|\mu_n + \mu_n^{v \otimes \bar{v}}\| \leq \|\zeta_n + (v \otimes \bar{v}) \zeta_n\|_2 \|\zeta'_n\|_2 \leq 2\|\zeta_n\|_2 \|\zeta'_n\|_2 = 2\|\mu_n\|,$$

one also has  $\|\zeta_n - (v \otimes \bar{v}) \zeta_n\| \rightarrow 0$  for every  $v \in \mathcal{U}(P)$ . Now, fix an ultralimit  $\text{Lim}$  and define  $\omega$  on  $\mathbb{B}(L^2(P))$  by  $\omega(x) = \text{Lim} \langle (x \otimes \bar{1}) \zeta_n, \zeta_n \rangle$ . Then  $\omega$  is an  $\mathcal{N}(P)$ -invariant  $P$ -central positive linear functional satisfying

$$\omega(p) = \text{Lim}_n |\mu_n|(p \otimes \bar{1}) \geq \text{Lim}_n |\mu_n(p \otimes \bar{1})| = \tau(p)$$

for every central projection  $p$  in  $P$ . By Lemma 5, we are done.  $\square$

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